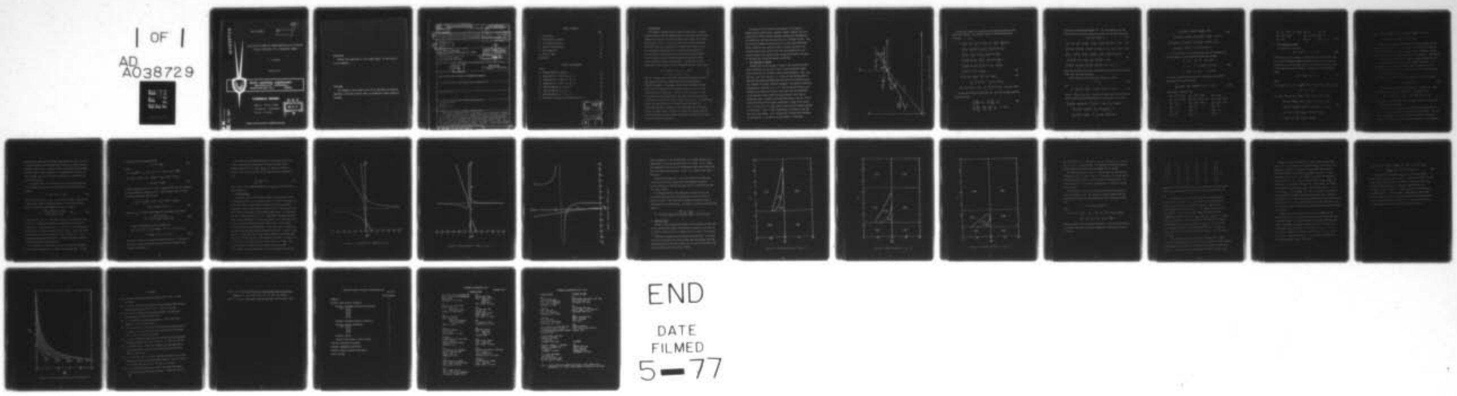
AD-A038 729 WATERVLIET ARSENAL N Y F/G 16/4
QUALITATIVE STABILITY CHARACTERISTICS OF A FLEXIBLE MISSILE SUB--ETC(U)
FEB 77 G L ANDERSON
UNCLASSIFIED WVT-TR-77007 NL

| OF |
AD
A038729



END
DATE
FILMED
5-77

WVT-TR-77007

AD

(12)

# QUALITATIVE STABILITY CHARACTERISTICS OF A FLEXIBLE MISSILE SUBJECTED TO A CIRCULATORY THRUST

G. L. Anderson

February 1977

US ARMY ARMAMENT COMMAND

**BENET WEAPONS LABORATORY**
**WATERVLIET ARSENAL**
**WATERVLIET, N.Y. 12189**

## TECHNICAL REPORT

AMCMS No. 611102.11.35D00

DA Project No. 1F161102A35D

Pron No. EJ-5-Y0015



APPROVED FOR PUBLIC RELEASE; DISTRIBUTION UNLIMITED

DISPOSITION

Destroy this report when it is no longer needed. Do not return it to the originator.

DISCLAIMER

The findings in this report are not to be construed as an official Department of the Army position unless so designated by other authorized documents.

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

14

## REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: WVT-TR-77007

2. GOVT ACCESSION NO.

3. RECIPIENT'S CATALOG NUMBER

9

4. TITLE (and Subtitle): Qualitative Stability Characteristics of a Flexible Missile Subjected to a Circulatory Thrust.

6

5. TYPE OF REPORT & PERIOD COVERED: Technical rept.

6. PERFORMING ORG. REPORT NUMBER

7. AUTHOR(s): G. L. Anderson

10

8. CONTRACT OR GRANT NUMBER(s)

16

9. PERFORMING ORGANIZATION NAME AND ADDRESS:
Benet Weapons Laboratory
Watervliet Arsenal, Watervliet, N.Y. 12189
SARWV-RT-TP

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS:
AMCMS No. 611102.11.35D00
DA Proj. No. 1F161102A35D
Pron No. EJ-5-Y0015

11. CONTROLLING OFFICE NAME AND ADDRESS:
US Army Armament Command
Rock Island, Illinois 61201

11

12. REPORT DATE: Feb [redacted] 77

13. NUMBER OF PAGES: 29

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)

12 28p.

15. SECURITY CLASS. (of this report): UNCLASSIFIED

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Circulatory Thrust
Dynamics
Flexible Missile
Stability

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The stability characteristics of an elastic, flexible missile of constant mass, modelled mathematically as a system consisting of three rigid bars elastically hinged at the joints and subjected to a constant circulatory thrust, are examined. Plots of eigencurves are presented and used to identify the regions of stability, divergence, and flutter in related stability maps. For anti-tangential and sub-tangential thrusts, the system is found to be divergent for any sufficiently small, positive thrust. Moreover, flutter does not occur for anti-tangential and super-tangential thrusts.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

## TABLE OF CONTENTS



## LIST OF ILLUSTRATIONS

## 1. INTRODUCTION

The problem of determining the state of stability of a slender, flexible missile, modeled mathematically as a free-free elastic beam subjected to a thrust, has been the subject of several recent investigations [1] - [9]. It is of particular interest that Wu [9] has determined that, in the absence of a feedback directional control, the mechanism of instability of the missile system described in references [2], [4], and [8] is not flutter but rather divergence. This discrepancy appears to result from the form of approximate solution assumed in those investigations. For example, to solve approximately the fundamental non-self-adjoint boundary value problem

$$y^{IV} + Q(xy')' - \omega^2 y = 0, \quad 0<x<1, \tag{1}$$

$$y'' = y''' = 0 \quad \text{at} \quad x = 0,1, \tag{2}$$

where $y' = dy/dx$, etc., and $Q$ and $\omega$ denote the dimensionless thrust and frequency parameters, respectively, Beal [2] employed the Galerkin procedure. He assumed a solution that included the rigid body motions of translation and rotation, and, as a result, always obtained two zero frequencies. However, it is very easy to verify that equations (1) and (2) admit a rigid body mode of rotation whenever $Q>0$, i.e., there exists only a single zero frequency. Subsequently, Wu [9] reported that the loss of stability of the nature of a structural mode that degenerates to the rigid body mode of rotation when $Q \to 0$ rather than by the onset of flutter which ensues upon the coalescence of two non-null natural frequencies at a positive critical value of the thrust parameter $Q$.

The objective of the present investigation is to examine a related missile problem from a somewhat simpler viewpoint, so as to expose rather clearly through elementary mathematical relationships the qualitative stability characteristics of a flexible missile. The system will be modeled mathematically as a system comprising three rigid, weightless bars joined by elastic springs, carrying a concentrated mass at the center of each bar, and subjected to a circulatory thrust, whose direction may be adjusted through a variation in the value of a so-called tangency coefficient.

## 2. THE EQUATIONS OF MOTION

Consider the system depicted in Figure 1 that consists of three rigid, weightless bars of equal length $\ell$ with elastically hinged joints that exert the following linear restoring moments: $c(\phi_1 - \phi_2)$ and $c(\phi_3 - \phi_2)$, where the $\phi_j$'s, $j = 1,2,3$, are the (assumed small) angles formed between the horizontal and each of the respective bars and c denotes a constant stiffness parameter. The neutral equilibrium configuration corresponds to $\phi_j = 0$. The concentrated masses $m_j$ are affixed at the center of each bar, and x and y are the coordinates of the position of the central mass $m_2$ relative to the fixed xy - coordinate frame. A thrust of constant magnitude $T_o$ with its line of action specified by the tangency coefficient $\alpha$, whose vector form $\underset{\sim}{F}$ is $\underset{\sim}{F} = T_o (\cos \phi_1 \underset{\sim}{i} + \sin \alpha\phi_1 \underset{\sim}{j})$, where $\underset{\sim}{i}$ and $\underset{\sim}{j}$ are unit vectors parallel to the fixed x - and y - axes, respectively, is applied at the left free end of the system. It is assumed that the system can translate in the vertical y - as well as the horizontal x - directions.

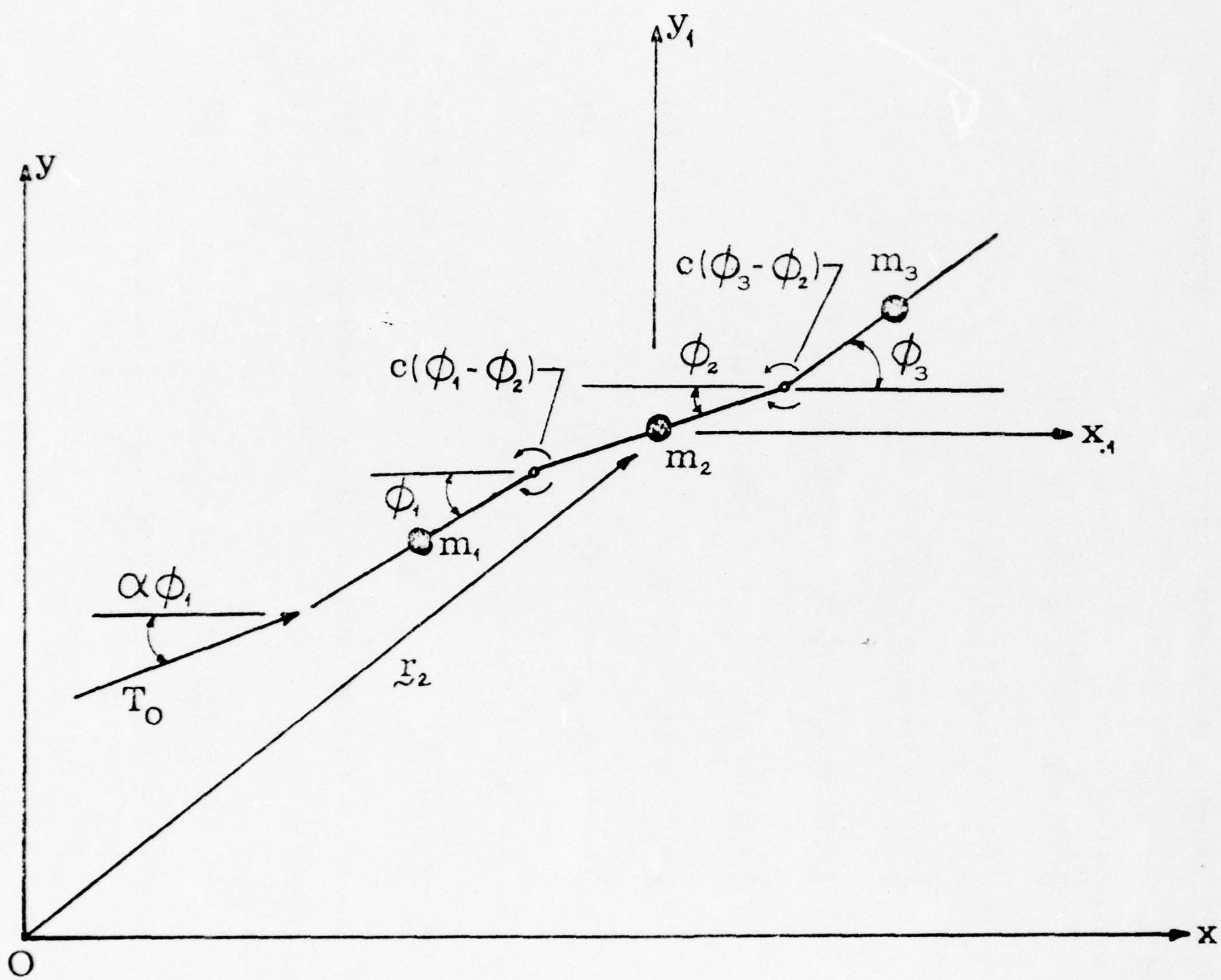



Figure 1. A simple model of a missile.

In the usual manner, one can derive the following expressions for the kinetic energy T, the potential energy V, and the generalized forces $Q_x$, . . ., $Q_3$:

$$T = \frac{1}{2}(m_1 + m_2 + m_3)\dot{x}^2 + \frac{1}{2}(m_1 + m_2 + m_3)\dot{y}^2 + \frac{1}{8}m_1\ell^2\dot{\phi}_1^2 +$$

$$+ \frac{1}{8}(m_1 + m_3)\ell^2\dot{\phi}_2^2 + \frac{1}{8}m_3\ell^2\dot{\phi}_3^2 + \frac{1}{2}m_1\ell\dot{x}\sin\phi_1\dot{\phi}_1 +$$

$$+ \frac{1}{2}(m_1 - m_3)\ell\dot{x}\sin\phi_2\dot{\phi}_2 - \frac{1}{2}m_3\ell\dot{x}\sin\phi_3\dot{\phi}_3 -$$

$$- \frac{1}{2}m_1\ell\dot{y}\cos\phi_1\dot{\phi}_1 - \frac{1}{2}(m_1 - m_3)\ell\dot{y}\cos\phi_2\dot{\phi}_2 +$$

$$+ \frac{1}{2}m_3\ell\dot{y}\cos\phi_3\dot{\phi}_3 + \frac{1}{4}m_1\ell^2\cos(\phi_2 - \phi_1)\dot{\phi}_1\dot{\phi}_2 +$$

$$+ \frac{1}{4}m_3\ell^2\cos(\phi_2 - \phi_3)\dot{\phi}_2\dot{\phi}_3 , \tag{3}$$

$$V = \frac{1}{2}c(\phi_1^2 - 2\phi_1\phi_2 + 2\phi_2^2 + \phi_3^2 - 2\phi_2\phi_3) , \tag{4}$$

$$Q_x = T_o\cos\alpha\phi_1, \quad Q_y = T_o\sin\alpha\phi_1,$$

$$Q_1 = T_o\ell\sin(1 - \alpha)\phi_1 , \quad Q_2 = \frac{1}{2}T_o\ell\sin(\phi_2 - \alpha\phi_1), \; Q_3 = 0. \tag{5}$$

In view of the forms of equations (3) - (5), the Lagrange equations may be written as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{x}}\right) = Q_x , \quad \frac{d}{dt}\left(\frac{\partial T}{\partial\dot{y}}\right) = Q_y , \tag{6}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\phi}_j}\right) - \frac{\partial T}{\partial\phi_j} + \frac{\partial V}{\partial\phi_j} = Q_j , \quad j = 1,2,3$$

Therefore, substituting equations (3) - (5) into equations (6), one finds that the linearized equations of motion for the given system are

$$(m_1 + m_2 + m_3)\ddot{x} = T_0 \tag{7}$$

$$(m_1 + m_2 + m_3)\ddot{y} - \frac{1}{2} m_1 \ell \ddot{\phi}_1 - \frac{1}{2}(m_1 - m_3)\ell\ddot{\phi}_2 + \frac{1}{2} m_3 \ell \ddot{\phi}_3 = \alpha T_0 \phi_1 \ , \tag{8}$$

$$\frac{1}{4} m_1 \ell^2 \ddot{\phi}_1 + \frac{1}{2} m_1 \ell \ddot{x} \phi_1 - \frac{1}{2} m_1 \ell \ddot{y} + \frac{1}{4} m_1 \ell^2 \ddot{\phi}_2 + c\,(\phi_1 - \phi_2) = T_0 \ell (1 - \alpha)\phi_1 ,$$

$$\frac{1}{4}(m_1 + m_3)\ell^2\ddot{\phi}_2 + \frac{1}{2}(m_1 - m_3)\ell\ddot{x}\phi_2 - \frac{1}{2}(m_1 - m_3)\ell\ddot{y} + \frac{1}{4} m_1 \ell^2 \ddot{\phi}_1 +$$

$$+ \frac{1}{4} m_3 \ell^2 \ddot{\phi}_3 - c\phi_1 + 2c\phi_2 - c\phi_3 = \frac{1}{2} T_0 \ell (\phi_2 - \alpha\phi_1) \ , \tag{9}$$

$$\frac{1}{4} m_3 \ell^2 \ddot{\phi}_3 - \frac{1}{2} m_3 \ell \ddot{x} \phi_3 + \frac{1}{2} m_3 \ell \ddot{y} + \frac{1}{4} m_3 \ell^2 \ddot{\phi}_2 + c\phi_3 - c\phi_2 = 0 \ ,$$

where the familiar small angle approximations have been made and higher order terms have been neglected.

Solving equations (7) and (8) for $\ddot{x}$ and $\ddot{y}$, one finds

$$\ddot{x} = \mu T_0 \tag{10}$$

and

$$\ddot{y} = \mu[\frac{1}{2} m_1 \ell \ddot{\phi}_1 + \frac{1}{2}(m_1 - m_3)\ell\ddot{\phi}_2 - \frac{1}{2} m_3 \ell \ddot{\phi}_3 + \alpha T_0 \phi_1] \ , \tag{11}$$

where $\mu = 1/(m_1 + m_2 + m_3)$. Insertion of equations (10) and (11) into equation (9) yields the following set of three equations of motion:

$$\frac{1}{4}\mu m_1 (m_2 + m_3)\ell^2\ddot{\phi}_1 + [c - \frac{1}{2}\mu T_0 \ell (1 - \alpha)(m_1 + 2m_2 + 2m_3)]\phi_1 +$$

$$+ \frac{1}{4}\mu m_1 (m_2 + 2m_3)\ell^2\ddot{\phi}_2 - c\phi_2 + \frac{1}{4}\mu m_1 m_3 \ell^2 \ddot{\phi}_3 = 0 \ ,$$

$$\frac{1}{4}\mu m_1 \ell^2 (m_2 + 2m_3)\ddot{\phi}_1 - [c - \frac{1}{2}\mu\alpha(m_2 + 2m_3)T_0 \ell]\phi_1 +$$

$$+ \frac{1}{4}\mu\ell^2(m_1m_2 + 4m_1m_3 + m_2m_3)\ddot{\phi}_2 + [2c -$$

$$- \frac{1}{2}\mu T_o\ell(m_2 + 2m_3)]\phi_2 + \frac{1}{4}\mu m_3\ell^2(2m_1 + m_2)\ddot{\phi}_3 - c\phi_3 = 0 , \tag{12}$$

$$\frac{1}{4}\mu m_1m_3\ell^2\ddot{\phi}_1 + \frac{1}{2}\mu m_3\alpha T_o\ell\phi_1 + \frac{1}{4}\mu m_3(2m_1 + m_2)\ell^2\ddot{\phi}_2 - d\phi_2 +$$

$$+ \frac{1}{4}\mu m_3(m_1 + m_2)\ell^2\ddot{\phi}_3 + (c - \frac{1}{2}\mu m_3T_o\ell)\phi_3 = 0 .$$

It is convenient to use a dimensionless form of equations (12). For this purpose, the following definitions may be made:

$$m_1 = \mu_1 m, \quad m_2 = m, \quad m_3 = \mu_3 m,$$
$$t = \sigma\tau, \quad \sigma^2 = m\ell^2/4c(1 + \mu_1 + \mu_3) , \tag{13}$$
$$Q = q/(1 + \mu_1 + \mu_3) , \quad q = T_o\ell/2c .$$

It is now a straightforward process to verify that the system of equations (12) assumes the form

$$\sum_{n=1}^{3} [Amn\ddot{\phi}n + (Cmn + QDmn)\phi n] = 0, \quad m = 1,2,3 \tag{14}$$

where now $\ddot{\phi}n = d^2\phi n/d\tau^2$, and

$A_{11} = \mu_1(1 + \mu_3)$, $A_{12} = \mu_1(1 + 2\mu_3)$, $A_{13} = \mu_1\mu_3$,

$A_{21} = \mu_1(1 + 2\mu_3)$, $A_{22} = \mu_1 + 4\mu_1\mu_3 + \mu_3$, $A_{23} = \mu_3(1 + 2\mu_1)$,

$A_{31} = \mu_1\mu_3$ , $A_{32} = \mu_3(1 + 2\mu_1)$, $A_{33} = \mu_3(1 + \mu_1)$,

$C_{11} = 1$, $C_{12} = - 1$, $C_{13} = 0$,

$C_{21} = - 1$, $C_{22} = 2$, $C_{23} = - 1$,

$C_{31} = 0$, $C_{32} = - 1$, $C_{33} = 1$,

$$D_{11} = (\alpha - 1)(2 + \mu_1 + 2\mu_3), \quad D_{12} = 0, \quad D_{13} = 0,$$
$$D_{21} = \alpha(1 + 2\mu_3), \quad D_{22} = -(1+2\mu_3), \quad D_{23} = 0,$$
$$D_{31} = \alpha\mu_3 \quad D_{32} = 0, \quad D_{33} = -\mu_3 . \tag{15}$$

## 3. THE FREQUENCY EQUATION

If a solution of the form

$$\phi_n(\tau) = X_n e^{i\omega\tau}, \quad n = 1,2,3 \tag{16}$$

where the $X_n$'s are constants, is substituted into equation (14), then a system of homogeneous algebraic equations in the $X_n$'s is obtained. This system has a non-trivial solution if and only if the determinant of the coefficient matrix vanishes. Expansion of this determinant yields

$$p_o\,\omega^4 - p_2\omega^2 + p_4 = 0, \tag{17}$$

where

$$p_o = 8\mu_1\mu_3(1 + \mu_1 + \mu_3) - \mu_1\mu_3 Q[3 + 4(\mu_1 + 2\mu_3) + (\mu_1 + \mu_3)(\mu_1 + 5\mu_3) - \\ - \alpha(1 + \mu_1 + \mu_3)^2],$$

$$\begin{aligned} p_2 = {} & 4(\mu_1 + 4\mu_1\mu_3 + \mu_3) - Q[3\mu_1 + 11\mu_3 + 36\mu_1\mu_3 + \mu_1^2 + 13\mu_3^2 + \\ & + 34\mu_1\mu_3^2 + 10\mu_1^2\mu_3 - \alpha(\mu_1 + 11\mu_3 + \mu_1^2 + 22\mu_1\mu_3 + 11\mu_3^2 + \\ & + 10\mu_1^2\mu_3 + 10\mu_1\mu_3^2)] + 2\mu_3 Q^2\,[1 + 3\mu_1 + 4\mu_3 + 11\mu_1\mu_3 + \\ & + \mu_1^2 + 3\mu_3^2 + 7\mu_1\mu_3^2 + 3\mu_1^2\mu_3 - \alpha(1 + 2\mu_1 + 4\mu_3 + \\ & + 7\mu_1\mu_3 + \mu_1^2 + 3\mu_3^2 + 3\mu_1\mu_3^2 + 3\mu_1^2\mu_3)], \end{aligned} \tag{18}$$

$$p_4 = -Q(1-\alpha)[\mu_3(2 + \mu_1 + 6\mu_3 + 2\mu_1\mu_3 + 4\mu_3^2)Q^2 - (2 + \mu_1 + 11\mu_3 + 4\mu_1\mu_3 + 10\mu_3^2)Q + 3 + \mu_1 + 5\mu_3] .$$

A priori, one would have expected the frequency equation (17) to be a bi-cubic polynomial in $\omega^2$. However, one can easily verify that the inertia matrix $\underset{\sim}{A} = (A_{mn})$ is singular which, therefore, accounts for the fact that the frequency equation is a biquadratic. Consequently, the system under consideration is a quasi - dynamic system [10], which is characterized by the existence of an internal constraint. To determine this constraint, one merely adds the equations for m = 1 and 3 obtained from equation (14) and then subtracts the equation for m = 2 from the result to obtain

$$[2 - Q\alpha(1 + \mu_3) - Q(1-\alpha)(2 + \mu_1 + 2\mu_3)]\phi_1 - [4 - Q(1 + 2\mu_3)]\phi_2 + (2 - \mu_3 Q)\phi_3 = 0 . \quad (19)$$

It is obvious from equation (18) that $p_4 = 0$ when Q - 0. Thus, the frequency equation (14) leads to $\omega^2 = 0$, $p_2^*/p_o^*$ , where

$$p_o^* = 8\mu_1\mu_3(1 + \mu_1 + \mu_3), \qquad p_2^* = 4(\mu_1 + 4\mu_1\mu_3 + \mu_3) .$$

The value $\omega^2 = 0$ corresponds to a rigid body rotation of the system.

## 4. STABILITY CONSIDERATIONS

If Q has an arbitrarily small positive value, then $p_4 \neq 0$ and $\omega^2 = 0$ is no longer a root of the frequency equation (17). Thus, one must conclude that the system can no longer undergo a rigid body rotation. As the value of Q is increased above Q = 0, an eigencurve

emanating from the origin in the $Q\omega^2$-plane must enter either the first or second quadrant, which implies that the system is either stable or unstable, respectively. Clearly, if one can establish a relationship between Q and $\omega^2$ that is invalid in a neighborhood of the origin in the $Q\omega^2$-plane, then some conclusions regarding the stability of the system can be drawn.

To accomplish this for small thrusts Q, the frequency parameter $\omega^2$ is expanded as a power series in the perturbation parameter Q as follows:

$$\omega^2 = Q\alpha_1 + Q^2\alpha_2 + \quad (Q^3), \tag{20}$$

where the coefficients $\alpha_1, \alpha_2, \ldots$ are to be determined. Substituting equation (20) into the frequency equation (17) and following the steps of the classical perturbation process, one finds

$$\omega^2 = \frac{-\,Q(1-\alpha)(3+\mu_1+5\mu_3)}{4(\mu_1+4\mu_1\mu_3+\mu_3)} + \quad (Q^2)\,, \tag{21}$$

provided that $0 \leq Q << 1$. From this equation it is evident that the sign of $\omega^2$ is determined solely by the magnitude of the tangency coefficient $\alpha$. Specifically, if $\alpha < 1$, then $\omega^2 < 0$ and the system is divergent for an arbitrarily small positive thrust. Hence, the value of the critical thrust of divergence is $Q_b = 0$ when $\alpha < 1$. If $\alpha > 1$ (a super-tangential thrust), then $\omega^2 > 0$ and the system is stable for sufficiently small, positive thrusts.

If the thrust is a tangential force ($\alpha = 1$), then, provided that $Q > 0$, $p_4$ again vanishes, as is evident from equation (18). In this

case, the frequency equation yields

$$\omega^2 = 0, \quad p_2/p_o \, , \tag{22}$$

where

$$p_o = 2\mu_1\mu_3[4(1 + \mu_1 + \mu_3) - Q(1 + \mu_1 + 3\mu_3 + 2\mu_1\mu_3 + 2\mu_3^2)] \, ,$$

$$p_2 = 4(\mu_1 + 4\mu_1\mu_3 + \mu_3) - 2Q[\mu_1(1 + 7\mu_3) + \mu_3^2(1 + 12\mu_1)] +$$

$$+ 2\mu_1\mu_3(1 + 2\mu_3)^2Q^2. \tag{23}$$

A change in the sign of $\omega^2$, as given in equation (22), will be signalled by the vanishing of either $p_2$ or $p_o$. Specifically, if $p_2 = 0$, then, according to equation (23) one has

$$\mu_1\mu_3(1 + 2\mu_3)^2Q_b^2 - [\mu_1(1 + 7\mu_3) + \mu_3^2(1 + 12\mu_1)Q_b +$$

$$+ 2(\mu_1 + 4\mu_1\mu_3 + \mu_3) = 0 \, , \tag{24}$$

whereas, if $p_o = 0$ the critical thrust for divergence is given by

$$Q_b = \frac{4(1 + \mu_1 + \mu_3)}{1 + \mu_1 + 3\mu_3 + 2\mu_1\mu_3 + 2\mu_3^2} \, . \tag{25}$$

It must be noted that $p_4$ can vanish whenever

$$\mu_3(2 + \mu_1 + 6\mu_3 + 2\mu_1\mu_3 + 4\mu_3^2)Q_b^2 - (2 + \mu_1 + 11\mu_3 + 4\mu_1\mu_3 +$$

$$+ 10\mu_3^2)Q_b + 3 + \mu_1 + 5\mu_3 = 0 \, , \tag{26}$$

which also provides a critical thrust for divergence. The values of $Q_b$ obtained from equation (26) are completely independent of the tangency coefficient $\alpha$.

The condition for the determination of the critical thrust for flutter, namely that of coalescence of a pair of roots of the frequency equation (17), is $p_2^2 - 4p_o p_4 = 0$ which, by virtue of equation (18), can be expressed as the following quartic polynomial in Q:

$$\sum_{n=1}^{5} r_n Q_e^{n-1} = 0, \tag{27}$$

where the $r_n$'s are lengthy polynomials in $\mu_1, \mu_3$, and $\alpha$ which will not be recorded here.

## 5. SOME EIGENCURVES

To identify regions of stability, flutter, and divergence in the stability maps that will be presented later, it is first efficacious to solve the frequency equation (17) and then to examine some typical eigencurves, such as those shown in Figures 2 to 4. These figures were prepared for three values of the tangency coefficient: $\alpha = 3/10$, $4/5$, and $3/2$, in the case of $\mu_1 = \mu_3 = 1$. In agreement with the conclusion drawn from equation (21), one sees from Figures 2 and 3 that the branch of the eigencurve emanating from the point $(\omega^2, Q) = (0,0)$ extends immediately into the second quadrant. Thus, the system is unstable by divergence for arbitrarily small, positive values of Q. Indeed, in the case of $\alpha = 3/10$, the value of the critical thrust of divergence must be $Q_b = 0$. In the case of $\alpha = 4/5$ (Figure 3), one has $\omega^2 = 0$ at $Q = 0$, $\omega^2 < 0$ for $0 < Q < 0.4127$, and $\omega^2 > 0$ for $0.4127 < Q < 0.506$. The bounds in these inequalities are obtained from equation (26). Consequently, one may conclude that the system diverges whenever the

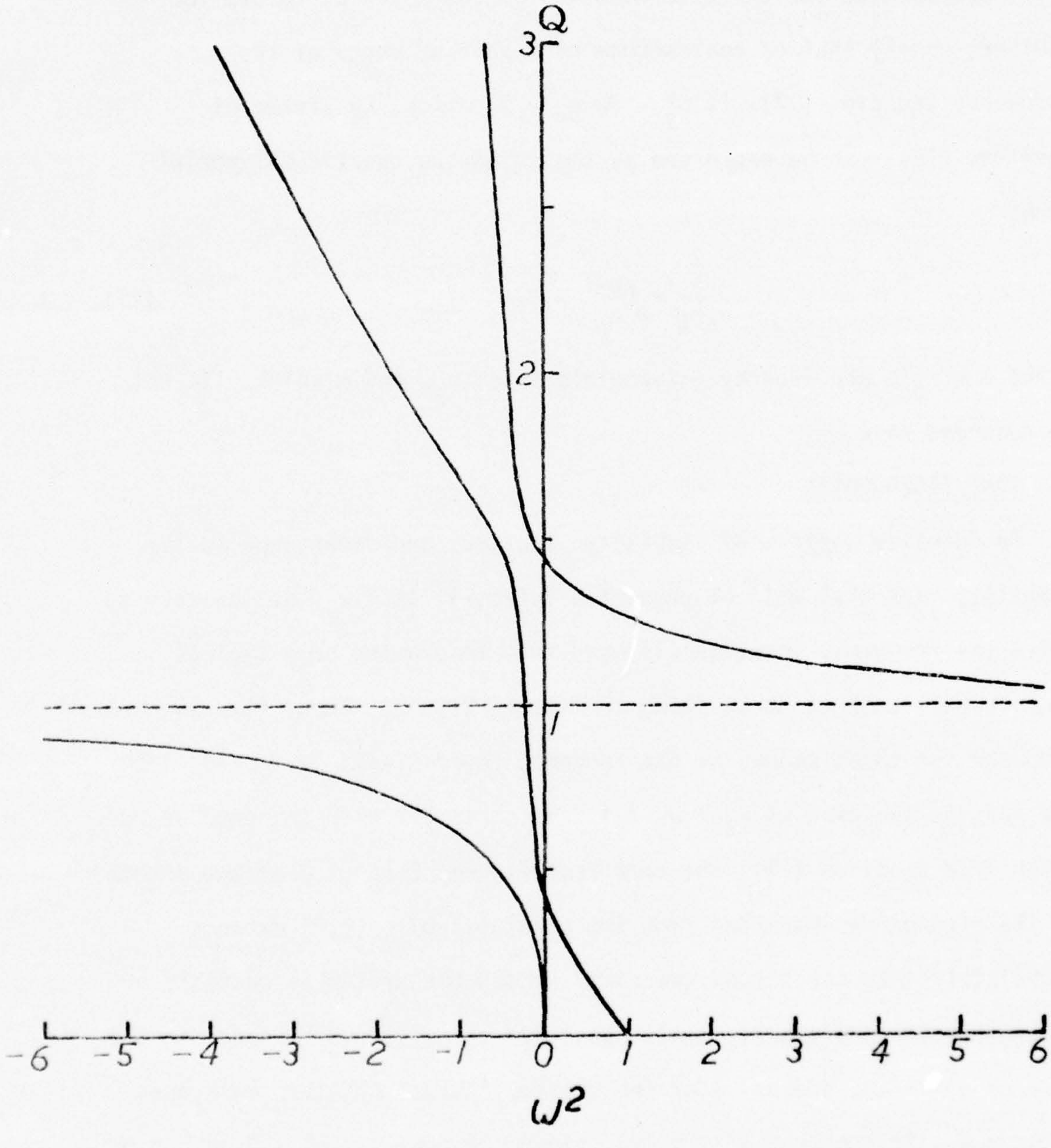



Figure 2. Eigencurve for $\alpha = 3/10$, $\mu_1 = \mu_3 = 1$.

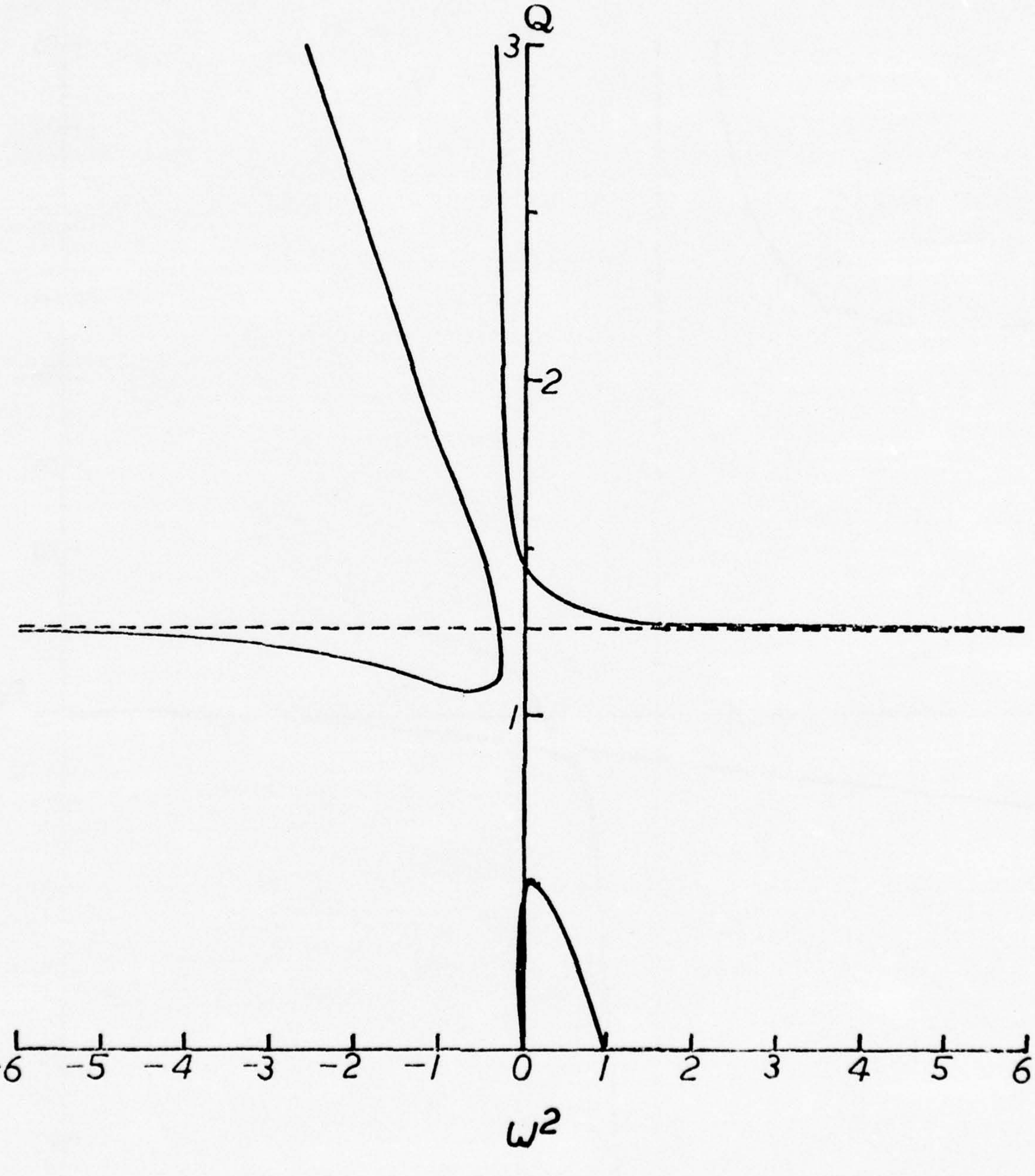



Figure 3. Eigencurve for $\alpha = 4/5$, $\mu_1 = \mu_3 = 1$.

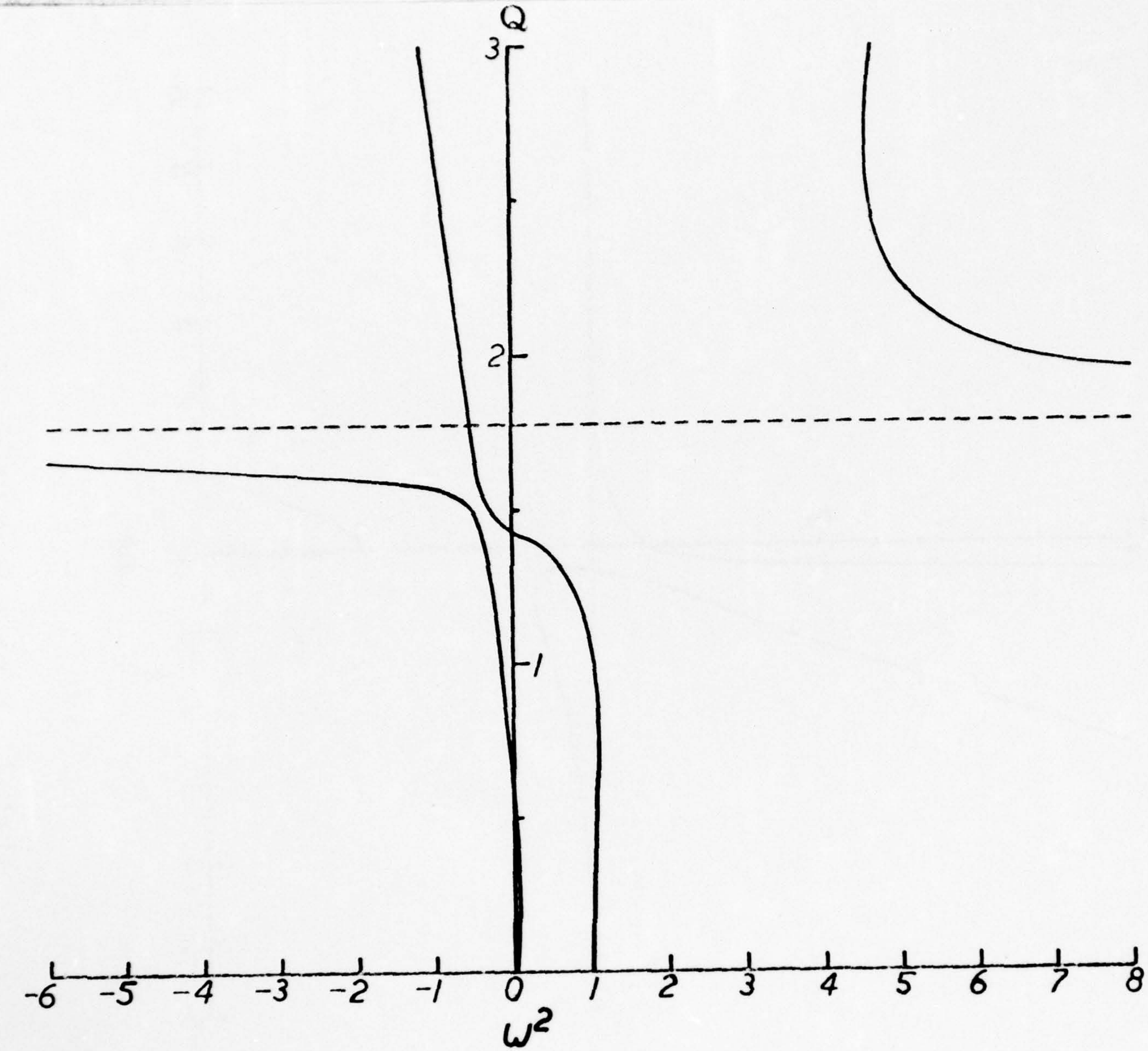



Figure 4. Eigencurve for $\alpha = 3/2$, $\mu_1 = \mu_3 = 1$.

thrust parameter Q is in the interval $0 < Q < 0.4127$ and that it is stable when Q is in the interval $0.4127 < Q < 0.506$. At $Q = 0.506$, the frequencies of the first and second modes merge, which signals the onset of instability by flutter. For $Q > 1.2$, flutter gives way to divergence.

In Figure 4 plotted for $\alpha = 3/2$, the branch of the eigencurve initiating at the origin enters the first quadrant, reaches a maximum value of $\omega^2$, and then decreases until it intersects the load axis at $Q_b = 0.4127$.

The dashed horizontal lines appearing in Figures 2 to 4 are horizontal asymptotes which represent the value of Q at which $\omega^2$ becomes infinite. This value of Q, hereafter designated as $Q_a$, is determined from the condition $p_o = 0$, which, by virtue of equation (18), leads to

$$Q_a = \frac{8(1 + \mu_1 + \mu_3)}{3 + 4(\mu_1 + 2\mu_3) + (\mu_1 + \mu_3)(\mu_1 + 5\mu_3) - \alpha(1 + \mu_1 + \mu_3)^2} .$$

## 6. STABILITY MAPS

Having in mind the observations made in the preceding section, one now can identify the regions of stability, divergence, and flutter in $q\alpha$-plane.* In Figures 5 to 7, stability maps are shown in the $q\alpha$-plane

---

*The thrust parameter q was defined in equation (13). It is more desirable to consider the $q\alpha$-plane rather than the $Q\alpha$-plane since the quantity Q tends to mask the influence of the mass distribution of the system on the value of the critical thrust.

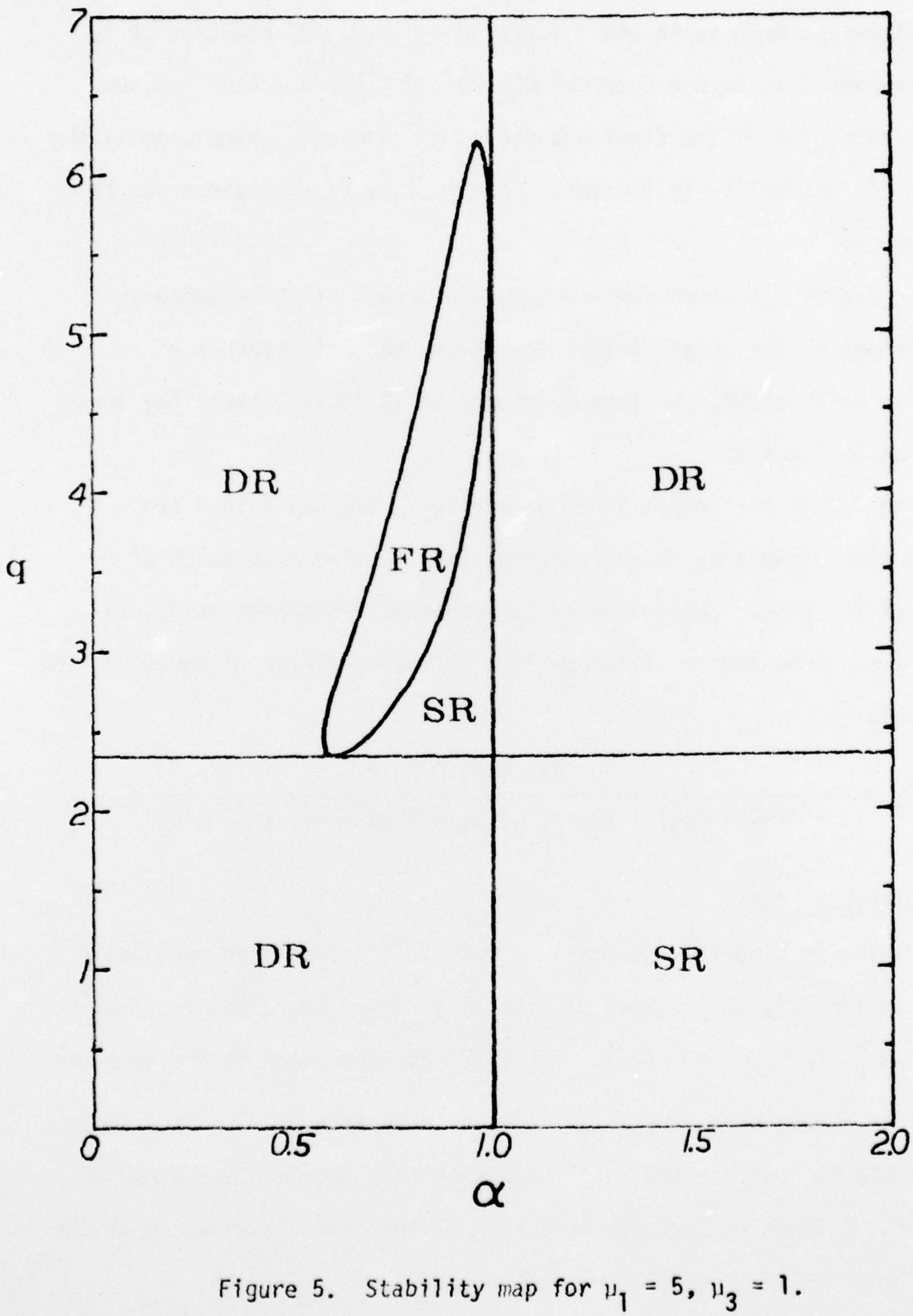



Figure 5. Stability map for $\mu_1 = 5$, $\mu_3 = 1$.

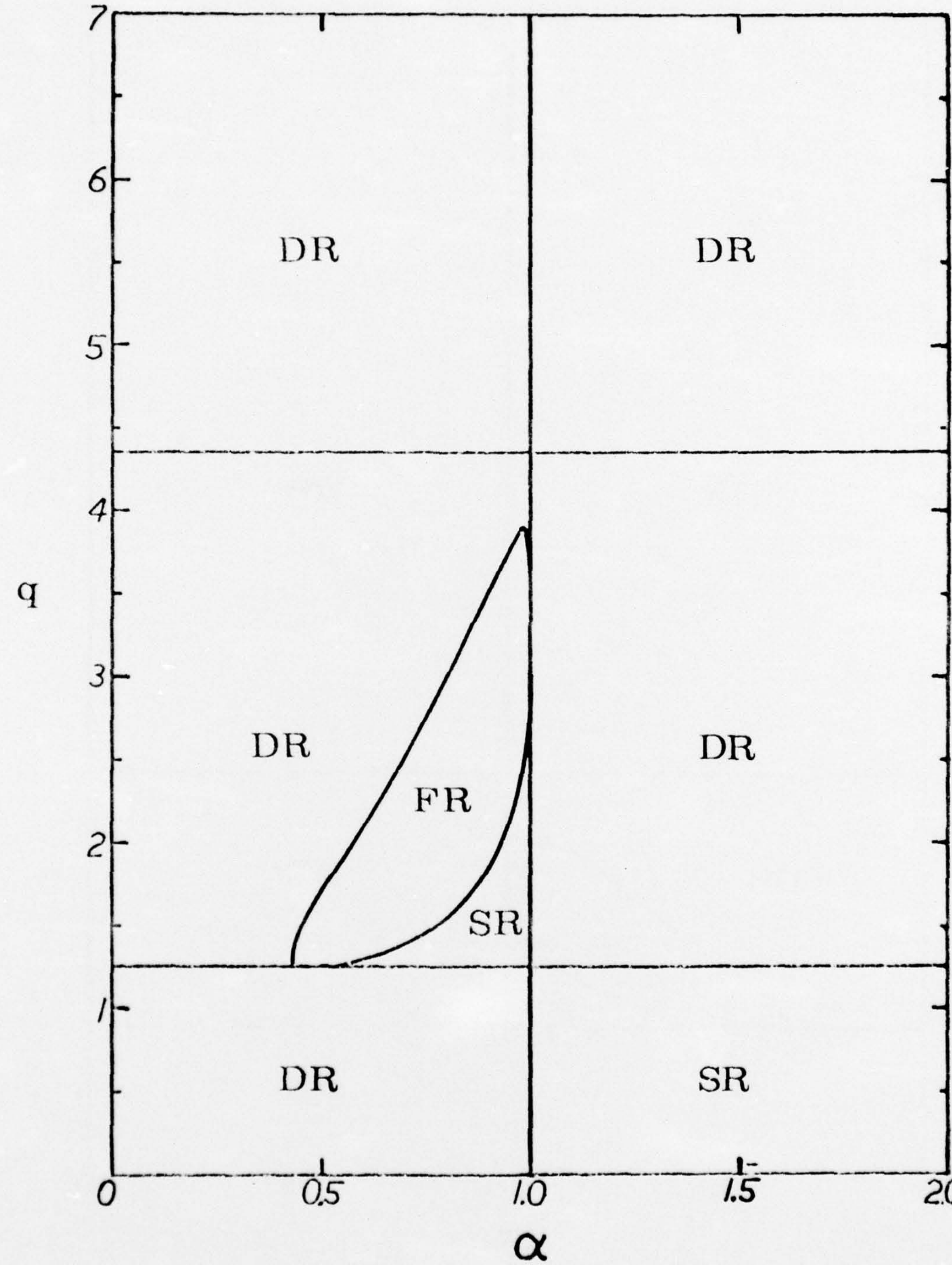



Figure 6. Stability map for $\mu_1 = \mu_3 = 1$.

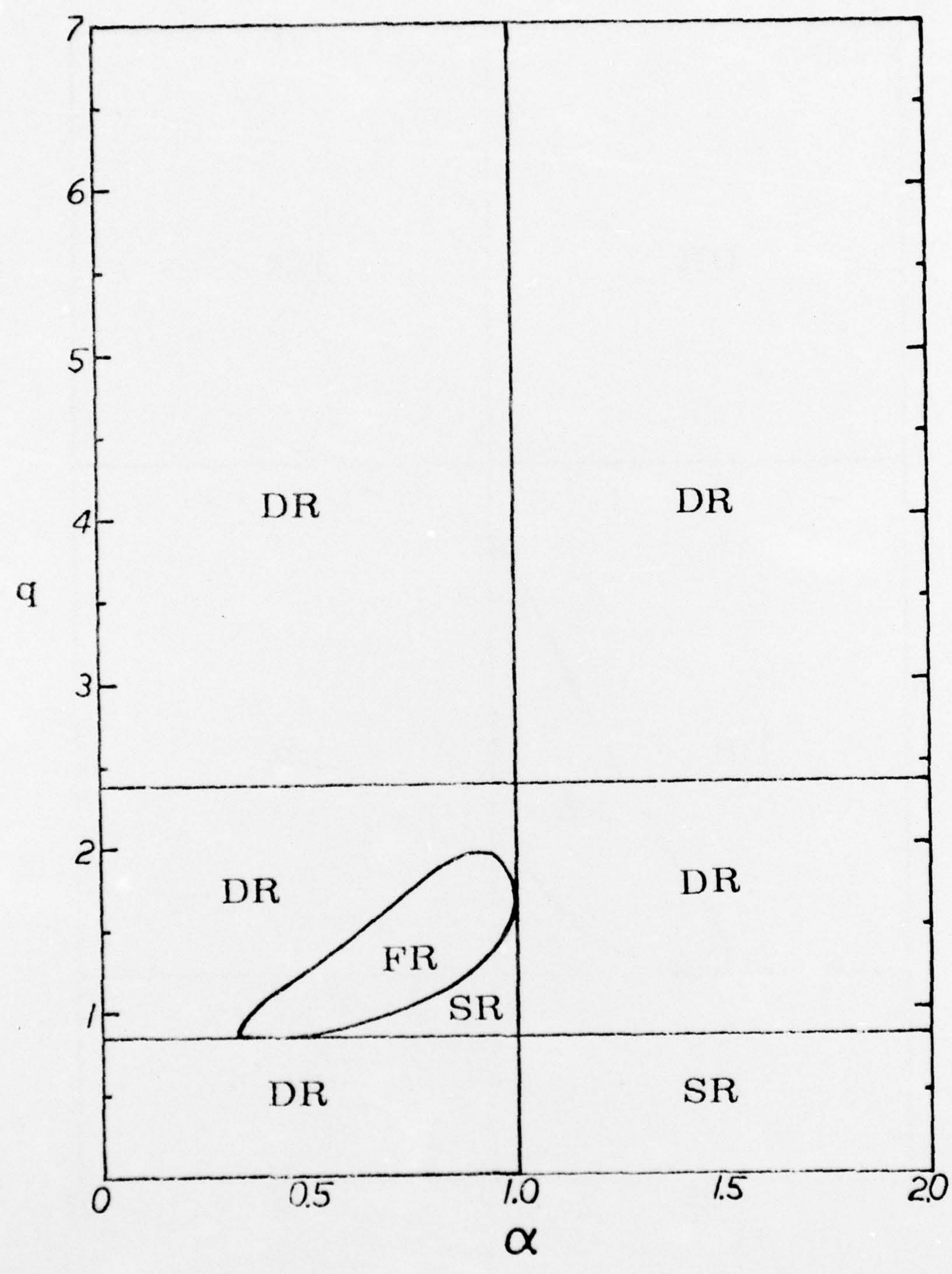



Figure 7. Stability map for $\mu_1 = 1$, $\mu_3 = 5$.

over the range $0 \leq \alpha \leq 2$ for (a) $\mu_1 = 5$, $\mu_3 = 1$, (b) $\mu_1 = \mu_3 = 1$, and (c) $\mu_1 = 1$, $\mu_3 = 5$, respectively. The regions of stability, flutter, and divergence are labelled with the symbols SR, FR, and DR.

It should be noted that, for $\alpha < 1$, the minimum critical thrust of divergence is $q_b = 0$, whereas, for $\alpha > 1$, its value is positive and is associated with the smallest of the two roots of equation (26). For anti-tangential ($\alpha < 0$) and super-tangential ($\alpha > 1$) thrusts, flutter does not occur.

From Figures 5 to 7, it is clear that there exists a value of $\alpha$, say $\alpha_*$, at which $q_b = q_e$. The value of $\alpha_*$ may be computed from the requirement that equation (27) and equation (26), written now in the form

$$b_1 + b_2 Q_b + b_3 Q_b^2 = 0 \ ,$$

where

$$b_1 = 3 + \mu_1 + 5\mu_3, \quad b_2 = -(2 + \mu_1 + 11\mu_3 + 4\mu_1\mu_3 + 10\mu_3^2),$$

$$b_3 = \mu_3(2 + \mu_1 + 6\mu_3 + 2\mu_1\mu_3 + 4\mu_3^2),$$

possess a common root in the thrust parameter. This requirement leads, according to Sylvester's dialytic method of elimination [11], to the condition

$$
\begin{vmatrix}
r_5 & r_4 & r_3 & r_2 & r_1 & 0 \\
0 & r_5 & r_4 & r_3 & r_2 & r_1 \\
b_3 & b_2 & b_1 & 0 & 0 & 0 \\
0 & b_3 & b_2 & b_1 & 0 & 0 \\
0 & 0 & b_3 & b_2 & b_1 & 0 \\
0 & 0 & 0 & b_3 & b_2 & b_1
\end{vmatrix} = 0 .
$$

Expansion of this determinant produces a polynomial in $\alpha$ from which the value of $\alpha_\star$ may be determined numerically for given values of $\mu_1$ and $\mu_3$.

Consequently, one may now observe that for a sub-tangential thrust ($\alpha < 1$) for which $0 < \alpha < \alpha_\star$, flutter cannot occur and the minimum critical thrust of divergence is $q_b = 0$. However, for a sub-tangential thrust for which $\alpha_\star < \alpha < 1$, flutter can occur (e.g., see Figure 3). For $\alpha$ in this latter range, the system is first unstable by divergence for $0 < q < q_{b_1}$, where $q_{b_1} = Q_{b_1}(1 + \mu_1 + \mu_3)$, with $Q_{b_1}$ denoting the smallest root of equation (26), and it becomes stable whenever $q_{b_1} < q < q_{e_1}$, where $q_{e_1} = Q_{e_1}(1 + \mu_1 + \mu_3)$, with $Q_{e_1}$ denoting the smallest real root of equation (27). For $q_{e_1} < q < q_{e_2}$, the system is prone to flutter, and for $q_{e_2} < q$ it is divergent once again.

The stability maps in Figures 5 to 7 indicate that a stable flight at relatively low values of the thrust parameter q can be achieved provided that the direction of the thrust is controlled in such a way that $\alpha > 1$. The critical super-tangential thrust is, of course, $q_{b_1}$.

Suppose next that the value of $\alpha$ is only slightly greater than unity. If, as the value of q is increased to $q_{b_1}$, the value of $\alpha$ is changed to a value very slightly less than unity, then a stable motion with thrusts greater than $q_{b_1}$ would be possible. In this second region of stability, the maximum permissible sub-tangential thrust would be $q < q_{e_1} < q^{\star}_{b_1}$, where the values of $q_{e_1}$ and $q^{\star}_{b_1}$ can be obtained from equations (27) and (24), respectively. For $\alpha$ only slightly less than unity, $q_{e_1} \approx q^{\star}_{b_1}$. For example, with $\mu_1 = \mu_3 = 1$, one finds $q_{b_1} = 1.2381$ and $q^{\star}_{b_1} = 3$. Thus, a significant increase in the admissible magnitude of the thrust appears possible, provided, certainly, that the control system can accomplish the necessary reduction in the value of $\alpha$ in a period of time that is sufficiently brief so as to avoid large angular changes relative to the equilibrium configuration of the system.

Finally, it is of interest to consider the influence of the distribution of mass of the system on the value of the critical super-tangential thrust. To do this, one may first presume that the total mass of the system is assigned some value. Then, in terms of the dimensionless mass parameters, one may write $1 + \mu_1 + \mu_3 = \beta + 1$, where $\beta$ is assigned a fixed value. Therefore, $\mu_1 = \beta - \mu_3$, where $0 \lesssim \mu_3 \lesssim \beta$. Substitution of $\mu_1 = \beta - \mu_3$ into equation (26) and introduction of $Q_b = q_b/(\beta + 1)$ lead to

$$\mu_3[2 + \beta + (5 + 2\beta)\mu_3 + 2\mu_3^2]q_b^2 - (\beta + 1)[2 + \beta + 2(5 + 2\beta)\mu_3 + 6\mu_3^2]q_b + (\beta + 1)^2(3 + \beta + 4\mu_3) = 0 \ . \qquad (28)$$

In Figure 8, the variation of $q_{b_1}$, i.e., the lowest root of equation (28), versus $\mu_3$ for $\beta = 1,2,\ldots,5$ has been plotted. All the curves show essentially the same behavior, namely, the value of $q_b$ decreases monotonically as $\mu_3$ increases, with the difference $\Delta = q_b(\beta') - q_b(\beta'')$, where $\beta' < \beta''$ and $\mu_3$ held fixed, increasing as the difference $\beta'' - \beta'$ is increased. Consequently, significantly greater critical thrusts result when the tail portion of the system is more massive than the head portion. The lowest critical thrusts are obtained when the head is more massive than the tail.

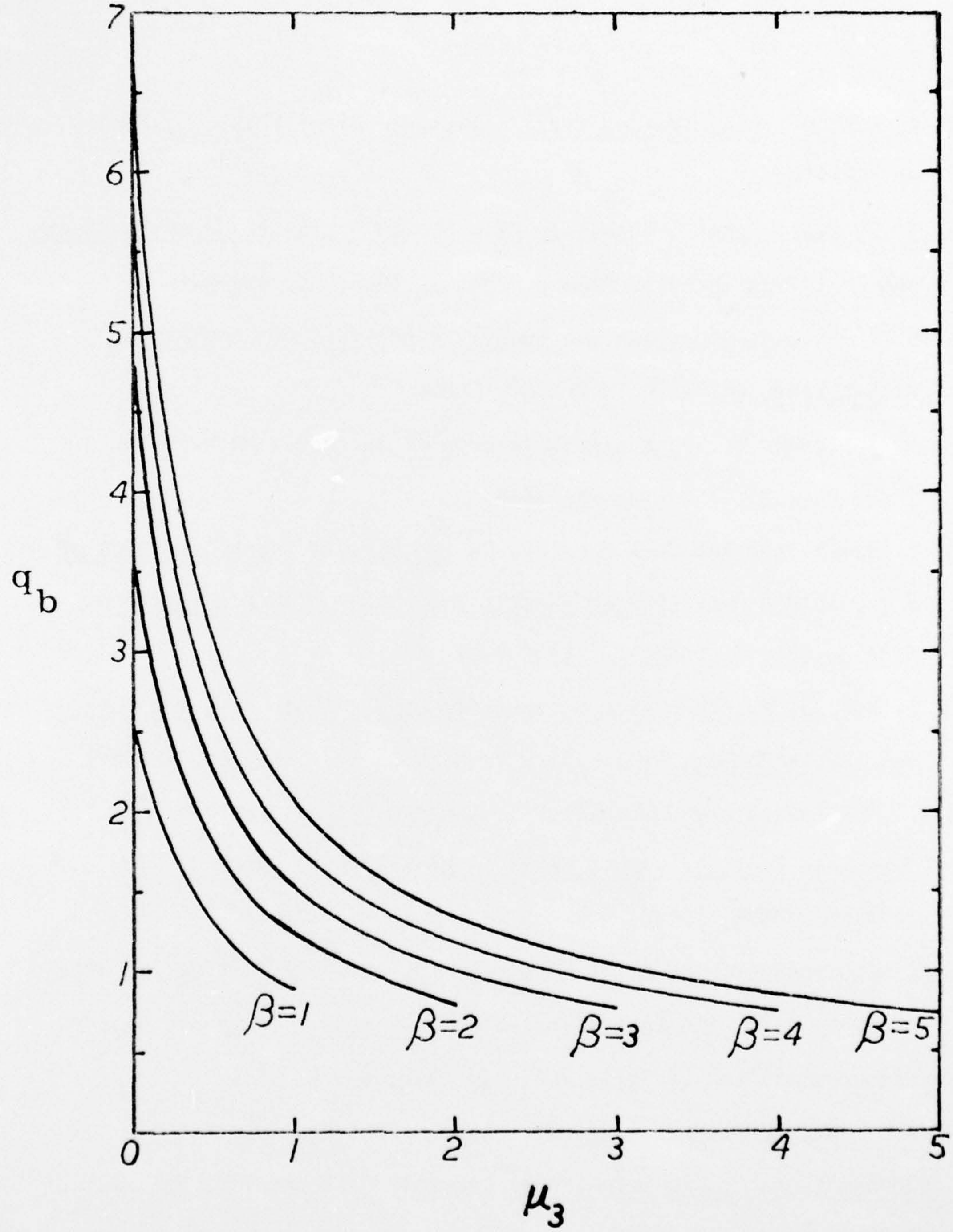



Figure 8. Variation of $q_b$ with $\mu_3$ for five values of $\beta$.

## REFERENCES


[1] P. Santini, Dinamica dei razzi polistadi, Missili Vol. 2, 1964, pp. 83-104.

[2] T. R. Beal, Dynamic Stability of a Flexible Missile Under Constant and Pulsating Thrusts, AIAA J. Vol. 3, 1965, pp. 486-494.

[3] R. F. Glaser, Vibration and Stability Analysis of Compressed Rocket Vehicles, NASA TN D-2533, 1965.

[4] V. I. Feodos'ev, On a Stability Problem, J. Appl. Math. Mech. (PMM) Vol. 29, 1965, pp. 445-446.

[5] L. Meirovitch and D. A. Wesley, On the Dynamic Characteristics of a Variable - Mass Slender Elastic Body Under High Accelerations, AIAA J. Vol. 5, 1967, pp. 1439-1447.

[6] L. Solarz, The Mechanism of the Loss of Stability of a Non-Guided Deformable Rocket, Proc. Vibr. Prob. Vol. 10, 1969, pp. 425-441.

[7] L. Meirovitch and J. Bankovskis, Dynamic Characteristics of a Two-Stage Variable - Mass Flexible Missile with Internal Flow, NASA CR - 2076, 1972.

[8] G. Y. Matsumoto and C. D. Mote, Time Delay Instabilities in Large Order Systems with Controlled Follower Forces. J. Dynamic Systems, Measurements and Control, Vol. 94, 1972, pp. 330-334.

[9] J. J. Wu, On the Stability of a Free - Free Beam Under Axial Thrust Subjected to Directional Control, J. Sound Vib. Vol. 43, 1975.

[10] A. B. Ku, On the Stability of a Linear Nongyroscopic Conservative System, Zeit. angew. Math. Phys. Vol. 20, 1971, pp. 986-991.

[11] W. V. Lovitt, Elementary Theory of Equations, Prentice-Hall, 1939.

WATERVLIET ARSENAL INTERNAL DISTRIBUTION LIST

May 1976

| | No. of Copies |
|---|---|
| COMMANDER | 1 |
| DIRECTOR, BENET WEAPONS LABORATORY | 1 |
| DIRECTOR, DEVELOPMENT ENGINEERING DIRECTORATE | 1 |
| ATTN: RD-AT | 1 |
| RD-MR | 1 |
| RD-PE | 1 |
| RD-RM | 1 |
| RD-SE | 1 |
| RD-SP | 1 |
| DIRECTOR, ENGINEERING SUPPORT DIRECTORATE | 1 |
| DIRECTOR, RESEARCH DIRECTORATE | 2 |
| ATTN: RR-AM | 1 |
| RR-C | 1 |
| RR-ME | 1 |
| RR-PS | 1 |
| TECHNICAL LIBRARY | 5 |
| TECHNICAL PUBLICATIONS & EDITING BRANCH | 2 |
| DIRECTOR, OPERATIONS DIRECTORATE | 1 |
| DIRECTOR, PROCUREMENT DIRECTORATE | 1 |
| DIRECTOR, PRODUCT ASSURANCE DIRECTORATE | 1 |
| PATENT ADVISORS | 1 |

EXTERNAL DISTRIBUTION LIST

December 1976

1 copy to each

OFC OF THE DIR. OF DEFENSE R&E
ATTN: ASST DIRECTOR MATERIALS
THE PENTAGON
WASHINGTON, D.C. 20315

CDR
US ARMY TANK-AUTMV COMD
ATTN: AMDTA-UL
AMSTA-RKM MAT LAB
WARREN, MICHIGAN 48090

CDR
PICATINNY ARSENAL
ATTN: SARPA-TS-S
SARPA-VP3 (PLASTICS TECH EVAL CEN)
DOVER, NJ 07801

CDR
FRANKFORD ARSENAL
ATTN: SARFA
PHILADELPHIA, PA 19137

DIRECTOR
US ARMY BALLISTIC RSCH LABS
ATTN: AMXBR-LB
ABERDEEN PROVING GROUND
MARYLAND 21005

CDR
US ARMY RSCH OFC (DURHAM)
BOX CM, DUKE STATION
ATTN: RDRD-IPL
DURHAM, NC 27706

CDR
WEST POINT MIL ACADEMY
ATTN: CHMN, MECH ENGR DEPT
WEST POINT, NY 10996

CDR
HQ, US ARMY AVN SCH
ATTN: OFC OF THE LIBRARIAN
FT RUCKER, ALABAMA 36362

CDR
US ARMY ARMT COMD
ATTN: AMSAR-PPW-IR
AMSAR-RD
AMSAR-RDG
ROCK ISLAND, IL 61201

CDR
US ARMY ARMT COMD
FLD SVC DIV
ARMCOM ARMT SYS OFC
ATTN: AMSAR-ASF
ROCK ISLAND, IL 61201

CDR
US ARMY ELCT COMD
FT MONMOUTH, NJ 07703

CDR
REDSTONE ARSENAL
ATTN: AMSMI-RRS
AMSMI-RSM
ALABAMA 35809

CDR
ROCK ISLAND ARSENAL
ATTN: SARRI-RDD
ROCK ISLAND, IL 61202

CDR
US ARMY FGN SCIENCE & TECH CEN
ATTN: AMXST-SD
220 7TH STREET N.E.
CHARLOTTESVILLE, VA 22901

DIRECTOR
US ARMY PDN EQ. AGENCY
ATTN: AMXPE-MT
ROCK ISLAND, IL 61201

EXTERNAL DISTRIBUTION LIST (Cont)

1 copy to each

CDR
US NAVAL WPNS LAB
CHIEF, MAT SCIENCE DIV
ATTN: MR. D. MALYEVAC
DAHLGREN, VA 22448

DIRECTOR
NAVAL RSCH LAB
ATTN: DIR. MECH DIV
WASHINGTON, D.C. 20375

DIRECTOR
NAVAL RSCH LAB
CODE 26-27 (DOCU LIB.)
WASHINGTON, D.C. 20375

NASA SCIENTIFIC & TECH INFO FAC
PO BOX 8757, ATTN: ACQ BR
BALTIMORE/WASHINGTON INTL AIRPORT
MARYLAND 21240

DEFENSE METALS INFO CEN
BATTELLE INSTITUTE
505 KING AVE
COLUMBUS, OHIO 43201

MANUEL E. PRADO / G. STISSER
LAWRENCE LIVERMORE LAB
PO BOX 808
LIVERMORE, CA 94550

DR. ROBERT QUATTRONE
CHIEF, MAT BR
US ARMY R&S GROUP, EUR
BOX 65, FPO N.Y. 09510

2 copies to each

CDR
US ARMY MOB EQUIP RSCH & DEV COMD
ATTN: TECH DOCU CEN
FT BELVOIR, VA 22060

CDR
US ARMY MAT RSCH AGCY
ATTN: AMXMR - TECH INFO CEN
WATERTOWN, MASS 02172

CDR
WRIGHT-PATTERSON AFB
ATTN: AFML/MXA
OHIO 45433

CDR
REDSTONE ARSENAL
ATTN: DOCU & TECH INFO BR
ALABAMA 35809

12 copies

CDR
DEFENSE DOCU CEN
ATTN: DDC-TCA
CAMERON STATION
ALEXANDRIA, VA 22314

NOTE: PLEASE NOTIFY CDR, WATERVLIET ARSENAL, ATTN: SARWV-RT-TP, WATERVLIET, N.Y. 12189, IF ANY CHANGE IS REQUIRED TO THE ABOVE.